

The strong uniform Artin-Rees property in codimension one

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1 Introduction

The purpose of this paper is to prove the following theorem:

Theorem 1 *Let A be an excellent (in fact $J - 2$) ring and let $N \subseteq M$ be two finitely generated A -modules such that $\dim(M/N) \leq 1$. Then there exists an integer $s \geq 1$ such that, for all integers $n \geq s$ and for all ideals I of A ,*

$$I^n M \cap N = I^{n-s}(I^s M \cap N).$$

This result is a variation of a theorem of Duncan and O’Carroll [DO]: maximal ideals are replaced for any ideal using the unavoidable hypothesis $\dim(M/N) \leq 1$ (as an Example of Wang shows [W₁]). Moreover it provides a partial positive answer to the question raised by Huneke in Conjecture 1.3 [H₁].

We begin by recalling what is called uniform Artin-Rees properties. Let A be a noetherian ring, I be an ideal of A and let $N \subseteq M$ be two finitely generated A -modules. The usual Artin-Rees lemma states that there exists an integer $s \geq 1$, depending on N , M and I , such that for all $n \geq s$,

$$I^n M \cap N = I^{n-s}(I^s M \cap N).$$

In particular, $I^n M \cap N \subseteq I^{n-s}N$. As in [H₁], let us say the pair (N, M) has the *(strong) uniform Artin-Rees property* with respect to a set of ideals \mathcal{W} of A and with *(strong) uniform number s* (s depending on $(N, M; \mathcal{W})$) if, for every ideal I of \mathcal{W} and for all $n \geq s$, $(I^n M \cap N = I^{n-s}(I^s M \cap N))$ $I^n M \cap N \subseteq I^{n-s}N$. Clearly, if s is a (strong) uniform number for (N, M, \mathcal{W}) and $t \geq s$, then t is also a (strong) uniform number for (N, M, \mathcal{W}) . The minimum of all such (strong) uniform numbers will be denoted by $s = s(N, M, \mathcal{W})$ and call it “the” (strong) uniform number for (N, M, \mathcal{W}) . If \mathcal{W} is the set of all ideals of A , we delete the phrase “with respect to \mathcal{W} ” and simply write $s = s(N, M)$.

Eisenbud and Hochster [EH] ask whether a pair (N, M) has the uniform Artin-Rees property with respect to the set of maximal ideals of A . O’Carroll [O₁] proves that if A is excellent then A has the uniform Artin-Rees property with respect to the set of maximal ideals and Duncan and O’Carroll [DO] generalize this result to the strong uniform Artin-Rees property. Later, O’Carroll [O₂] shows the strong uniform Artin-Rees property with respect to the set of principal ideals of a noetherian ring A . Nevertheless, the strong uniform Artin-Rees property cannot hold for the class of all ideals of A . Indeed, Wang [W₁] shows that if (A, \mathfrak{m}) is a 3-dimensional regular local ring,

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$\mathfrak{m} = (x, y, z)$, $I_k = (x^k, y^k, x^{k-1}y + z^k)$ and $J = (z)$, then there does not exist an $s \geq 1$ such that, for all $n \geq s$ and for all $k \geq 1$, $I_k^n \cap J = I_k^{n-s}(I_k^s \cap J)$. Remark that $\dim(A/J) = 2$. Thus, in this sense, Theorem 1 is not improvable.

On the other hand Huneke [H₁] shows the uniform Artin-Rees property with respect to the class of all ideals of a noetherian ring A if A is either essentially of finite type over a noetherian local ring, either a ring of characteristic p and a module finite over A^p or either essentially of finite type over \mathbb{Z} . In the same paper Huneke conjectures that this theorem remains true for excellent noetherian rings of finite Krull dimension. Thus Theorem 1 gives a partial positive answer to this conjecture.

Since strong uniform Artin-Rees property is not true in general, Huneke [H₂] asks for classes of ideals where strong uniform Artin-Rees property holds. If A is regular local, J is an ideal of A , does there exist an $s \geq 1$ such that, for all $n \geq s$ and for all ideal I of A whose image in A/J is generated by a system of parameters, $I^n \cap J = I^{n-s}(I^s \cap J)$? In fact, Lai [L] proves that this property is equivalent to the Relation-Type Conjecture, stated by Huneke, and proved by Wang [W₂] for rings with finite local cohomology. The *relation type* of an ideal I , $\text{rt}(I)$, is the largest degree of any minimal homogeneous system of generators of the ideal defining the Rees algebra of I . The Relation-Type Conjecture asks whether there is an integer $s \geq 1$ such that, for all parameter ideal I of a complete local equidimensional noetherian ring A , the relation type of I is $\text{rt}(I) \leq s$.

In order to prove Theorem 1 we generalise this relationship between the strong uniform Artin-Rees property and the existence of uniform bounds for the relation type. First we define $\text{rt}(I; M)$, the relation type of an ideal I with respect to an A -module M (Section 2). Then we consider $\text{grt}(M) = \sup\{\text{rt}(I; M) \mid I \text{ ideal of } A\}$, the supremum (possibly infinite) of all relation types of ideals I of A with respect to M , and call it the *global relation type* of the A -module M . We prove:

Theorem 2 *Let A be a commutative ring, \mathcal{W} a set of ideals of A , $I \in \mathcal{W}$ and $N \subseteq M$ two A -modules. Let $s(N, M; \mathcal{W})$ denote the strong uniform number for the pair (N, M) with respect to \mathcal{W} . Then $s(N, M; \{I\}) \leq \text{rt}(I; M/N) \leq \max(\text{rt}(I; M), s(N, M; \{I\}))$. In particular, $s(N, M; \mathcal{W}) \leq \sup\{\text{rt}(J; M/N) \mid J \in \mathcal{W}\}$ and $s(N, M) \leq \text{grt}(M/N)$.*

We thus ask for whether a module has finite global relation type. A very special case is already known: for a commutative (non necessarily noetherian local) domain A , $\text{grt}(A) = 1$ is equivalent to A be a ring of Prüfer [Cos] and, more in general, commutative rings with $\text{grt}(A) = 1$ are known to be the rings of weak dimension one or less [P₁]. Thus, for a noetherian local ring A , $\text{grt}(A) = 1$ if and only if A is a discrete valuation ring or a field.

Our guide here is the following celebrated theorem of Cohen and Sally [Coh], [S]: for a commutative noetherian local ring (A, \mathfrak{m}, k) , $\sup\{\mu(I) \mid I \text{ ideal of } A\} < \infty$ is equivalent to $\dim A \leq 1$, where $\mu(I) = \dim_k(I/\mathfrak{m}I)$ stands for the minimum number of generators of I . Then, we show the expected analogous result by replacing $\mu(I)$ for the relation type $\text{rt}(I)$ of I . Concretely:

Theorem 3 *Let A be an excellent (in fact $J - 2$) ring. The following conditions are equivalent:*

- (i) $\text{grt}(M) < \infty$ for all finitely generated A -module M .
- (ii) $\text{grt}(A) < \infty$.
- (iii) There exists an $r \geq 1$ such that $\text{rt}(I) \leq r$ for every three-generated ideal I of A .
- (iv) There exists an $r \geq 1$ such that $(x^r y)^r \in (x^{r+1}, y^{r+1})(x^{r+1}, y^{r+1}, x^r y)^{r-1}$ for all $x, y \in A$.
- (v) $\dim A \leq 1$.

The paper is organized as follows. Section 2 is dedicated to recall some definitions and properties on the module of effective relations of a graded algebra. In order to prove Theorem 2, we need to generalize them from graded algebras to graded modules. Once introduced all the machinery, we prove Theorem 2 in Section 3. In Section 4 we prove that rings of finite global relation type have dimension one or less and in Section 5 we show that zero dimensional modules over noetherian rings have finite global relation type. This is half of the proof in Theorem 3. In Section 6, we first consider the local case and reduce to Cohen-Macaulay modules. Then we give a new proof, now for modules, of a well known result for rings (see, for instance, [T₂]): if I is an \mathfrak{m} -primary ideal of a one dimensional Cohen-Macaulay local ring A and M is a maximal Cohen-Macaulay A -module, then $\text{rt}(I; M) \leq e(A)$, the relation type of I with respect M is bounded above by the multiplicity of A . We conclude that one dimensional finitely generated modules over noetherian local rings have finite global relation type. Section 7 finishes with all proofs. Throughout, A denotes a commutative ring with unity. All tensor products are over A unless specified the contrary. Dimension of a ring or module always mean Krull dimension. One of the main tools in this note is the module of effective relations of a graded algebra or module. In order to recall some of their general properties we will often refer to [P₂].

2 Preliminaries

Let A be a commutative ring. By a *standard* A -algebra we mean a commutative graded A -algebra $U = \bigoplus_{n \geq 0} U_n$ with $U_0 = A$ and such that U is generated as an A -algebra by the elements of U_1 . Put $U_+ = \bigoplus_{n > 0} U_n$ the *irrelevant ideal* of U .

If $E = \bigoplus_{n \geq 0} E_n$ is a graded U -module and $r \geq 0$ is an integer, we denote by $F_r(E)$ the submodule of E generated by the elements of degree at most r . Put (possibly infinite) $s(E) = \min\{r \geq 1 \mid E_n = 0 \text{ for all } n \geq r + 1\}$. Remark that we are only interested for $s(E) \geq 1$. Since for all $n \geq 1$, $(E/U_+E)_n = E_n/U_1E_{n-1}$, then for all $r \geq 1$, the following three conditions are equivalent: $F_r(E) = E$; $s(E/U_+E) \leq r$; and $E_n = U_1E_{n-1}$ for all $n \geq r + 1$.

If $f : V \rightarrow U$ is a surjective graded morphism of standard A -algebras, we denote by $E(f)$ the graded A -module $E(f) = \ker f / V_+ \ker f = \bigoplus_{n \geq 1} \ker f_n / V_1 \ker f_{n-1} = \bigoplus_{n \geq 1} E(f)_n$. The following is an elementary but very useful fact (Lemma 2.1 [P₂]): if $f : V \rightarrow U$ and $g : W \rightarrow V$ are two surjective graded morphisms of standard A -algebras, then there exists a graded exact sequence of A -modules $E(g) \rightarrow E(f \circ g) \xrightarrow{g} E(f) \rightarrow 0$. In particular, $s(E(f)) \leq s(E(f \circ g)) \leq \max(s(E(f)), s(E(g)))$. Moreover, if V and W are two symmetric algebras, then $E(g)_n = 0$ and $E(f \circ g)_n = E(f)_n$ for all $n \geq 2$.

Let U be a standard A -algebra, let $\mathbf{S}(U_1)$ be the symmetric algebra of U_1 and let $\alpha : \mathbf{S}(U_1) \rightarrow U$ be the surjective graded morphism of standard A -algebras induced by the identity on U_1 . The *module of effective n -relations* of U is defined to be $E(U)_n = E(\alpha)_n = \ker \alpha_n / U_1 \ker \alpha_{n-1}$ (for $n = 0, 1$, $E(U)_n = 0$). Put $E(U) = \bigoplus_{n \geq 2} E(U)_n = \bigoplus_{n \geq 2} E(\alpha)_n = E(\alpha) = \ker \alpha / \mathbf{S}_+(U_1) \ker \alpha$. The *relation type* of U is defined to be $\text{rt}(U) = s(E(U))$, that is, $\text{rt}(U)$ is the minimum positive integer $r \geq 1$ such that the effective n -relations are zero for all $n \geq r + 1$.

A *symmetric presentation* of U is a surjective graded morphism of standard A -algebras $f : V \rightarrow U$, where $V = \mathbf{S}(V_1)$ is the symmetric A -algebra of the A -module V_1 (for instance, $V_1 = U_1$ and $f_1 = 1$, or $f_1 : V_1 \rightarrow U_1$ a free presentation of U_1). Using Lemma 2.1 in [P₂] one deduces that $E(U)_n = E(f)_n$ for all $n \geq 2$ and $s(E(U)) = s(E(f))$. Thus the module of effective n -relations and the relation type of a standard A -algebra are independent of the chosen symmetric presentation.

For an ideal I of A , the module of effective n -relations and the relation type of I are defined to be $E(I)_n = E(\mathcal{R}(I))_n$ and $\text{rt}(I) = \text{rt}(\mathcal{R}(I))$, where $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n t^n \subset A[t]$ is the *Rees algebra* of I . If $\text{rt}(I) < \infty$ (for instance, if A is noetherian), then $\text{rt}(I) = \text{rt}(\mathcal{G}(I))$, where $\mathcal{G}(I) = \bigoplus_{n \geq 0} I^n / I^{n+1}$ is the *associated graded ring* of I (Proposition 3.3 [P₂]).

Let us now extend the classical notion of relation type of an ideal to the relation type of an ideal with respect to a module. Some of the results we present here are a straightforward generalization of former results. We thus will skip some details.

Definition 2.1 Let $U = \bigoplus_{n \geq 0} U_n$ be a standard A -algebra and $F = \bigoplus_{n \geq 0} F_n$ a graded U -module. We will say F is a *standard U -module* if F is generated as an U -module by the elements of F_0 , that is, $F_n = U_n F_0$ for all $n \geq 0$. In particular, $F_n = U_1 F_{n-1}$ for all $n \geq 1$.

Examples 2.2 Some of the most interesting standard modules for our purposes are the following:

- (1) If U is a standard A -algebra and M is an A -module, then $U \otimes M$ is a standard U -module (M in degree zero). If $U_1 = A^{\oplus n}$ is a finitely generated free A -module and $U = \mathbf{S}(U_1)$ is the symmetric algebra of U_1 , then $U \otimes M = A[T_1, \dots, T_n] \otimes M = M[T_1, \dots, T_n]$.
- (2) $\mathcal{R}(I; M) = \bigoplus_{n \geq 0} I^n M$, the *Rees module* of an ideal I of A with respect to an A -module M , is a standard $\mathcal{R}(I)$ -module.
- (3) $\mathcal{G}(I; M) = \bigoplus_{n \geq 0} I^n M / I^{n+1} M$, the *associated graded module* of an ideal I of A with respect to an A -module M , is a standard $\mathcal{G}(I)$ -module.

Let $U = \bigoplus_{n \geq 0} U_n$ be a standard A -algebra and $F = \bigoplus_{n \geq 0} F_n$, $G = \bigoplus_{n \geq 0} G_n$ be two graded U -modules. If $\varphi : G \rightarrow F$ is a surjective graded morphism of U -modules, we denote by $E(\varphi)$ the graded A -module $E(\varphi) = \ker \varphi / U_+ \ker \varphi = \ker \varphi_0 \oplus (\bigoplus_{n \geq 1} \ker \varphi_n / U_1 \ker \varphi_{n-1}) = \bigoplus_{n \geq 0} E(\varphi)_n$. The following is a generalization of Lemma 2.1 in [P₂]:

Lemma 2.3 If $\varphi : G \rightarrow F$ and $\psi : H \rightarrow G$ are two surjective graded morphisms of graded U -modules, then there exists a graded exact sequence $E(\psi) \rightarrow E(\varphi \circ \psi) \xrightarrow{\psi} E(\varphi) \rightarrow 0$ of A -modules. In particular, $s(E(\varphi)) \leq s(E(\varphi \circ \psi)) \leq \max(s(E(\varphi)), s(E(\psi)))$. Moreover, if $H = \mathbf{S}(P) \otimes Q$ is the tensor product of the symmetric algebra of the A -module P with the A -module Q , $G = \mathbf{S}(M) \otimes N$ is the tensor product of the symmetric algebra of the A -module M with the A -module N and $\psi = f \otimes h$ where $f : \mathbf{S}(P) \rightarrow \mathbf{S}(M)$ is induced by an epimorphism $f_1 : P \rightarrow M$ and $h : Q \rightarrow N$ is also surjective, then $E(\psi)_n = 0$ and $E(\varphi \circ \psi)_n = E(\varphi)_n$ for all $n \geq 2$.

Proof. To deduce the existence of the exact sequence we proceed as in Lemma 2.1 in [P₂]. For the second assertion, consider the following commutative diagram of exact rows:

$$\begin{array}{ccccccc}
 & & & & 1 \otimes \psi_{n-2} & & \\
 & & & & \Lambda_2(U_1) \otimes \mathbf{S}_{n-2}(P) \otimes Q \xrightarrow{\quad} \Lambda_2(U_1) \otimes \mathbf{S}_{n-2}(M) \otimes N \longrightarrow & 0 & \\
 & & & \partial_{2,n}^P \otimes 1_Q \downarrow & & \downarrow \partial_{2,n}^M \otimes 1_N & \\
 U_1 \otimes \ker \psi_{n-1} & \longrightarrow & U_1 \otimes \mathbf{S}_{n-1}(P) \otimes Q & \xrightarrow{1 \otimes \psi_{n-1}} & U_1 \otimes \mathbf{S}_{n-1}(M) \otimes N & \longrightarrow & 0 \\
 \downarrow & & \partial_{1,n}^P \otimes 1_Q \downarrow & & \downarrow \partial_{1,n}^M \otimes 1_N & & \\
 0 \longrightarrow & \ker \psi_n & \longrightarrow & \mathbf{S}_n(P) \otimes Q & \xrightarrow{\psi_n} & \mathbf{S}_n(M) \otimes N & \longrightarrow 0
 \end{array}$$

where $\partial_{2,n}^P((x \wedge y) \otimes z) = y \otimes xz - x \otimes yz$ and $\partial_{1,n}^P(x \otimes t) = xt$, $x, y \in U_1$, $z \in \mathbf{S}_{n-2}(P)$, $t \in \mathbf{S}_{n-1}(P)$ (∂^M defined analogously). By Theorem 2.4 in [P₂], the right and middle columns are exact sequences for all $n \geq 2$ and, by the snake lemma, $\ker(\partial_{1,n}^P \otimes 1_Q) \rightarrow \ker(\partial_{1,n}^M \otimes 1_N) \rightarrow E(\psi)_n \rightarrow 0$ are exact for all $n \geq 2$. Since $1 \otimes \psi_{n-2}$ is surjective, $E(\psi)_n = 0$ for all $n \geq 2$. ■

Definition 2.4 Let U be a standard A -algebra and F be a standard U -module. Let $\mathbf{S}(U_1)$ be the symmetric algebra of U_1 and let $\alpha : \mathbf{S}(U_1) \rightarrow U$ be the surjective graded morphism of standard A -algebras induced by the identity on U_1 . Let $\gamma : \mathbf{S}(U_1) \otimes F_0 \xrightarrow{\alpha \otimes 1} U \otimes F_0 \rightarrow F$ be the composition of $\alpha \otimes 1$ with the structural morphism. Since F is a standard U -module, γ is a surjective graded morphism of graded $\mathbf{S}(U_1)$ -modules. The *module of effective n -relations* of F is defined to be $E(F)_n = E(\gamma)_n = \ker \gamma_n / U_1 \ker \gamma_{n-1}$ (for $n = 0$, $E(F)_n = 0$). Put $E(F) = \bigoplus_{n \geq 1} E(F)_n = \bigoplus_{n \geq 1} E(\gamma)_n = E(\gamma) = \ker \gamma / \mathbf{S}_+(\mathbf{S}(U_1)) \ker \gamma$. The *relation type* of F is defined to be $\text{rt}(F) = s(E(F))$, that is, $\text{rt}(F)$ is the minimum positive integer $r \geq 1$ such that the effective n -relations are zero for all $n \geq r + 1$.

A *symmetric presentation* of a standard U -module F is a surjective graded morphism of standard V -modules $\varphi : G \rightarrow F$, with $\varphi : G = V \otimes M \xrightarrow{f \otimes h} U \otimes F_0 \rightarrow F$ where $f : V \rightarrow U$ is a symmetric presentation of the standard A -algebra U , $h : M \rightarrow F_0$ is an epimorphism of A -modules and $U \otimes F_0 \rightarrow F$ is the structural morphism. Using Lemma 2.3, one deduces that $E(F)_n = E(\varphi)_n$ for all $n \geq 2$ and $s(E(F)) = s(E(\varphi))$. Thus the module of effective n -relations and the relation type of a standard U -module are independent of the chosen symmetric presentation.

For an ideal I of A and an A -module M , the module of effective n -relations and the relation type of I with respect to M are defined to be $E(I; M)_n = E(\mathcal{R}(I; M))_n$ and $\text{rt}(I; M) = \text{rt}(\mathcal{R}(I; M))$.

Remark 2.5 The following are simple, but useful remarks:

- (1) If U is a standard A -algebra, then U is a standard U -module. Moreover the modules of effective n -relations of U as a standard A -algebra and as a standard U -module are equal $E_{A\text{-alg}}(U)_n = E_{U\text{-mod}}(U)_n$. Thus $\text{rt}_{A\text{-alg}}(U) = \text{rt}_{U\text{-mod}}(U)$. In particular, if I is an ideal of A , then $E(I; A)_n = E(I)_n$ and $\text{rt}(I; A) = \text{rt}(I)$.
- (2) If $f : V \rightarrow U$ is a surjective graded morphism of standard A -algebras and F is a standard U -module, then F is a standard V -module. Moreover $E_{U\text{-mod}}(F)_n = E_{V\text{-mod}}(F)_n$ for all $n \geq 2$ and $\text{rt}_{U\text{-mod}}(F) = \text{rt}_{V\text{-mod}}(F)$.
- (3) If $\varphi : R \rightarrow A$ is a surjective homomorphism of rings, U is standard A -algebra and F is a standard U -module, then $V = R \oplus U_+$ is a standard R -algebra and F is a standard V -module. Moreover $E(U)_n = E(V)_n$ for all $n \geq 2$ and $\text{rt}(U) = \text{rt}(V)$. Analogously $E_{U\text{-mod}}(F)_n = E_{V\text{-mod}}(F)_n$ for all $n \geq 2$ and $\text{rt}_{U\text{-mod}}(F) = \text{rt}_{V\text{-mod}}(F)$.
- (4) If $\varphi : G \rightarrow F$ is a surjective graded morphism of standard U -modules such that $\ker \varphi_n = 0$ for all $n \geq t$, then $E(G)_n = E(F)_n$ for all $n \geq t + 1$ and $\text{rt}(G) \leq \max(\text{rt}(F), t)$. If $t = 1$, then $\text{rt}(G) = \text{rt}(F)$. For instance, if I and J are two ideals of A , $\text{rt}(I/I \cap J) = \text{rt}((I + J)/J)$.
- (5) Let F be a standard U -module, $\underline{x} = \{x_i\}$ a (possibly infinite) set of generators of the A -module U_1 and $\underline{T} = \{T_i\}$ a set of as many variables over A as \underline{x} has elements. Take $V_1 = \bigoplus_i A T_i$, $V = \mathbf{S}(V_1) = A[\underline{T}]$, $G = V \otimes F_0 = F_0[\underline{T}]$ and $\varphi : G \rightarrow F$ defined by $\varphi(\sum y_i T_i) = \sum x_i y_i$. Clearly φ is a symmetric presentation of F . Thus, $\text{rt}(F) = 1$ if and only if $\ker \varphi$ is generated by linear forms. If $I = (\underline{x})$ is an ideal of A and M is an A -module, then $\text{rt}(I; M) = 1$ if and only

if the kernel of the surjective graded morphism $\varphi : M[\underline{T}] \rightarrow \mathcal{R}(I; M)$, $\varphi(\sum y_i T_i) = \sum x_i y_i$, is generated by linear forms. We say I is an *ideal of linear type with respect to M* if $\text{rt}(I; M) = 1$ ([HSV] pag 106, [T₁] pag 41).

Proof. (1) follows from definitions. (2) is consequence of Lemma 2.3. For the proof of (3), consider $\alpha_V : \mathbf{S}^R(U_1) \rightarrow V$ surjective graded morphism of standard R -algebras, $\alpha_U : \mathbf{S}^A(U_1) \rightarrow U$ surjective graded morphism of standard A -algebras, $f : V \rightarrow U$ and $g : \mathbf{S}^R(U_1) \rightarrow \mathbf{S}^A(U_1)$ the natural surjective graded morphisms extending φ . Since $f \circ \alpha_V = \alpha_U \circ g$ and f_n and g_n are isomorphisms for all $n \geq 1$, then $E(V)_n = E(\alpha_V)_n = E(f \circ \alpha)_n = E(\alpha_U)_n = E(U)_n$. For the rest of (3) is sufficient to apply the tensor product $- \otimes F_0$. In order to prove (4), let $\psi : H \rightarrow G$ be a symmetric presentation of G . By hypothesis, $\ker \psi_n = \ker(\varphi \circ \psi)_n$ for all $n \geq t$. Hence $E(G)_n = E(\psi)_n = E(\varphi \circ \psi)_n = E(F)_n$ for all $n \geq t + 1$. If $n \geq t + 1$, $\text{rt}(F) + 1$, then $E(G)_n = E(F)_n = 0$. Thus $\text{rt}(G) \leq \max(\text{rt}(F), t)$. Take $G = \mathcal{R}(I/I \cap J)$, $F = \mathcal{R}((I + J)/J)$ and $\varphi : G \rightarrow F$ the natural surjective graded morphism with $\varphi_0 : A/I \cap J \rightarrow A/J$ and $\varphi_n : (I^n + I \cap J)/(I \cap J) \xrightarrow{\sim} (I^n + J)/J$ isomorphism for all $n \geq 1$. Applying consecutively (1), (4), (3) and (1), $E_{A/I \cap J\text{-alg}}(G)_n = E_{G\text{-mod}}(G)_n = E_{G\text{-mod}}(F)_n = E_{A/J\text{-alg}}(F)_n$. Finally, (5) follows from definitions. ■

Let us now modify Theorem 2.4 in [P₂] to modules:

Proposition 2.6 *Let U be a standard A -algebra and let F be a standard U -module. For each integer $n \geq 2$, there exists a complex of A -modules*

$$\Lambda_2(U_1) \otimes F_{n-2} \xrightarrow{\partial_{2,n}} U_1 \otimes F_{n-1} \xrightarrow{\partial_{1,n}} F_n,$$

defined by $\partial_{2,n}((x \wedge y) \otimes z) = y \otimes xz - x \otimes yz$ and $\partial_{1,n}(x \otimes t) = xt$ and whose homology is $E(F)_n$.

Proof. By Theorem 2.4, there exists $\Lambda_2(U_1) \rightarrow U_1 \otimes U_1 \rightarrow U_2 \rightarrow 0$, a complex of A -modules defined by $\partial_2(x \wedge y) = y \otimes x - x \otimes y$ and $\partial_1(x \otimes t) = xt$. Applying the tensor product $- \otimes F_{n-2}$ and considering the structural morphisms $U_i \otimes F_j \rightarrow F_{i+j}$ we get the complex. Let $\mathbf{S}(U_1)$ be the symmetric algebra of U_1 and let $\alpha : \mathbf{S}(U_1) \rightarrow U$ be the surjective graded morphism of standard A -algebras induced by the identity on U_1 . Let $\gamma : \mathbf{S}(U_1) \otimes F_0 \xrightarrow{\alpha \otimes 1} U \otimes F_0 \rightarrow F$ be the composition of $\alpha \otimes 1$ with the structural morphism. Consider now, for each $n \geq 2$, the following commutative diagram of exact rows:

$$\begin{array}{ccccccc} & & & & 1 \otimes \gamma_{n-2} & & \\ & & & & \downarrow & & \\ & & \Lambda_2(U_1) \otimes \mathbf{S}_{n-2}(U_1) \otimes F_0 & \xrightarrow{\quad} & \Lambda_2(U_1) \otimes F_{n-2} & \longrightarrow & 0 \\ & & \downarrow \partial_{2,n}^S \otimes 1_{F_0} & & \downarrow & \partial_{2,n}^F & \\ U_1 \otimes \ker \gamma_{n-1} & \longrightarrow & U_1 \otimes \mathbf{S}_{n-1}(U_1) \otimes F_0 & \xrightarrow{1 \otimes \gamma_{n-1}} & U_1 \otimes F_{n-1} & \longrightarrow & 0 \\ & & \downarrow \partial_{1,n}^S \otimes 1_{F_0} & & \downarrow & \partial_{1,n}^F & \\ 0 & \longrightarrow & \ker \gamma_n & \longrightarrow & \mathbf{S}_n(U_1) \otimes F_0 & \xrightarrow{\gamma_n} & F_n \longrightarrow 0 \end{array}$$

By Theorem 2.4 in [P₂], the middle column is exact. Thus $\ker(\partial_{1,n}^S \otimes 1_{F_0}) = \text{im}(\partial_{2,n}^S \otimes 1_{F_0})$. Hence, $(1 \otimes \gamma_{n-1})(\ker(\partial_{1,n}^S \otimes 1_{F_0})) = \text{im}((1 \otimes \gamma_{n-1}) \circ (\partial_{2,n}^S \otimes 1_{F_0})) = \text{im}(\partial_{2,n}^F)$. Using the snake lemma, we conclude that $E(F)_n = \ker(\partial_{1,n}^F)/\text{im}(\partial_{2,n}^F)$. ■

Remark 2.7 As a corollary of Proposition 2.6 we have (see also 3.1, 3.2 and 3.3 in [P₂]):

- (1) Let U be a cyclic standard A -algebra generated by a degree one form $x \in U_1$. If F is a standard U -module, then $E(F)_n = (0 : x) \cap F_{n-1}$ and $\text{rt}(F) = \min\{r \geq 1 \mid (0 :_F x^{r+1}) = (0 :_F x^r)\}$. If $U = \mathcal{R}(I)$ is the Rees algebra of a principal ideal $I = (x)$ of A and $F = \mathcal{R}(I; M)$ is the Rees module of I with respect to a module M , then $E(I; M) = (0 : x) \cap I^{n-1}M$ and $\text{rt}(I; M) = \min\{r \geq 1 \mid (0 :_M x^{r+1}) = (0 :_M x^r)\}$.
- (2) If $\varphi : A \rightarrow B$ is a homomorphism of rings, U is standard A -algebra and F is a standard U -module, then $U \otimes B$ is a standard B -algebra and $F \otimes B$ is a standard $U \otimes B$ -module. Moreover $\text{rt}_{U \otimes B\text{-mod}}(F \otimes B) \leq \text{rt}_{U\text{-mod}}(F)$. If φ is flat, $E_{U \otimes B\text{-mod}}(F \otimes B) = E_{U\text{-mod}}(F) \otimes B$. In particular, $\text{rt}(F) = \sup\{\text{rt}(F_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Spec}(A)\} = \sup\{\text{rt}(F_{\mathfrak{m}}) \mid \mathfrak{m} \in \text{Max}(A)\}$.
- (3) If U is a standard A -algebra, F is a standard U -module and $J \subseteq \text{Ann}_A(F_0)$, then $U \otimes A/J$ is a standard A/J -algebra, $F \otimes A/J = F$ is a standard $U \otimes A/J$ -module, $E_{U\text{-mod}}(F)_n = E_{U \otimes A/J\text{-mod}}(F)_n$ and $\text{rt}_{U\text{-mod}}(F) = \text{rt}_{U \otimes A/J\text{-mod}}(F)$.
- (4) If $\text{rt}(I; M) < \infty$ (for instance, if A is noetherian and M is a finitely generated A -module), then $\text{rt}(I; M) = \text{rt}(\mathcal{G}(I; M))$. In particular, if $J \subset I$, then $\text{rt}(\mathcal{R}(I; M) \otimes A/J) = \text{rt}(I; M)$.

3 Proof of Theorem 2

Definition 3.1 Let $\text{grt}(M) = \sup\{\text{rt}(I; M) \mid I \text{ ideal of } A\}$ denote the supremum (possibly infinite) of all relation types of ideals I of A with respect to the A -module M , and let us call it the *global relation type* of M . Remark that:

- (1) If $M = A$, $\text{grt}(A) = \sup\{\text{rt}(I) \mid I \text{ ideal of } A\}$. We will prove that for an excellent ring A , $\text{grt}(A) < \infty$ is equivalent to $\dim A \leq 1$.
- (2) Since $\text{rt}(I; M) = \sup\{\text{rt}(I_{\mathfrak{p}}; M_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Spec}(A)\} = \sup\{\text{rt}(I_{\mathfrak{m}}; M_{\mathfrak{m}}) \mid \mathfrak{m} \in \text{Max}(A)\}$, then $\text{grt}(M) = \sup\{\text{grt}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Spec}(A)\} = \sup\{\text{grt}(M_{\mathfrak{m}}) \mid \mathfrak{m} \in \text{Max}(A)\}$.
- (3) If necessary to specify, we will write $\text{grt}(M) = \text{grt}_A(M)$ when considering M an A -module. For instance, if $J \subseteq \text{Ann}_A(M) = \{x \in A \mid xM = 0\}$, then $\mathcal{R}((I+J)/J; M) = \mathcal{R}(I; M)$. Thus $\text{rt}(I; M) = \text{rt}((I+J)/J; M)$ and $\text{grt}_A(M) = \text{grt}_{A/J}(M)$.

Theorem 2 Let A be a commutative ring, \mathcal{W} a set of ideals of A , $I \in \mathcal{W}$ and $N \subseteq M$ two A -modules. Let $s(N, M; \mathcal{W})$ denote the strong uniform number for the pair (N, M) with respect to \mathcal{W} . Then $s(N, M; \{I\}) \leq \text{rt}(I; M/N) \leq \max(\text{rt}(I; M), s(N, M; \{I\}))$. In particular, $s(N, M; \mathcal{W}) \leq \sup\{\text{rt}(J; M/N) \mid J \in \mathcal{W}\}$ and $s(N, M) \leq \text{grt}(M/N)$.

Proof. Let $F = \mathcal{R}(I; M/N)$, $G = \mathcal{R}(I; M)$, $H = \mathbf{S}(I) \otimes M$, $\varphi : G \rightarrow F$ the surjective graded morphism of standard $\mathbf{S}(I)$ -algebras defined by $\varphi_n : G_n = I^n M \rightarrow I^n M / I^n M \cap N = I^n M + N / N = F_n$ and $\gamma : H \rightarrow G$ induced by the natural graded morphism $\alpha : \mathbf{S}(I) \rightarrow \mathcal{R}(I)$. By Lemma 2.3, $s(E(\varphi)) \leq s(E(\varphi \circ \gamma)) \leq \max(s(E(\varphi)), s(E(\gamma)))$. But, $s(E(\varphi \circ \gamma)) = \text{rt}(I; M/N)$ and $s(E(\gamma)) = \text{rt}(I; M)$. Finally, since $E(\varphi)_n = (I^n M \cap N) / I(I^{n-1} M \cap N)$, then $s(E(\varphi)) = s(N, M; \{I\})$. In particular, $s(N, M; \{I\}) \leq \text{rt}(I; M/N) \leq \sup\{\text{rt}(J; M/N) \mid J \in \mathcal{W}\}$, and taking the supremum over all ideals I of \mathcal{W} , $s(N, M; \mathcal{W}) \leq \sup\{\text{rt}(J; M/N) \mid J \in \mathcal{W}\}$. ■

Corollary 3.2 ARTIN-REES LEMMA. Let A be a commutative ring, I an ideal of A and $N \subseteq M$ two A -modules. If $\text{rt}(I; M/N) < \infty$ then $s(N, M; \{I\}) < \infty$. In particular, if A is noetherian and

M is finitely generated, there exists an integer $s \geq 1$ such that, for all integers $n \geq s$, $I^n M \cap N = I^{n-s}(I^s M \cap N)$.

Corollary 3.3 O'CARROLL [O₁]. *Let A be a noetherian ring and let M be a finitely generated A -module. Then $\sup\{\text{rt}((x); M) \mid x \in A\} < \infty$. In particular, if $N \subset M$, there exists an integer $s \geq 1$ such that, for all integers $n \geq s$ and for all $x \in A$, $x^n M \cap N = x^{n-s}(x^s M \cap N)$.*

Proof. Following very closely the proof of O'Carroll in [O₁], let $0 = Q_1 \cap \dots \cap Q_r$ be a minimal primary decomposition of 0 in M , $r_M(Q_i) = r(Q_i : M) = \mathfrak{p}_i \in \text{Spec}(A)$, and let $s \geq 1$ be an integer such that, for all $i = 1, \dots, r$, $\mathfrak{p}_i^s M \subseteq Q_i$. Then, for all $x \in A$, $\text{rt}((x); M) \leq s$. Indeed, if $x \in \mathfrak{p}_i$, $x^{n+s} \in \mathfrak{p}_i^{n+s}$ and $(Q_i : x^{n+s}) = M$. If $x \notin \mathfrak{p}_i$, $x^{n+s} \notin \mathfrak{p}_i^{n+s}$ and $(Q_i : x^{n+s}) = Q_i$. Therefore, for all $n \geq 0$, $(0 : x^{n+s}) = (\cap_i Q_i : x^{n+s}) = \cap_i (Q_i : x^{n+s}) = \cap_{x \notin \mathfrak{p}_i} Q_i$. In particular, $(0 : x^{s+1}) = (0 : x^s)$ and $\text{rt}((x); M) \leq s$. We finish by applying Theorem 2. ■

Remark 3.4 Let A be a noetherian ring and let M be a finitely generated A -module. Let $\text{gr}^i(M) = \sup\{\text{rt}(I; M) \mid \mu(I) \leq i\}$. By 3.3, $\text{gr}^1(M) < \infty$. Using the example of Wang [W₁] and Theorem 2, we know $\text{gr}^3(M)$ might be infinite. We do not know whether $\text{gr}^2(M)$ is finite.

4 Rings of finite global relation type have dimension one

Remark 4.1 Let A be a commutative ring and let $r \geq 1$ denote an integer. Consider the following conditions:

- (a) $\text{rt}(I) \leq r$ for every three-generated ideal I of A .
- (b) $E(I)_{r+1} = 0$ for every three-generated ideal I of A .
- (c) $(x, y)(x, y, z)^r : z^{r+1} = (x, y)(x, y, z)^{r-1} : z^r$ for all $x, y, z \in A$.
- (d) $(x^r y)^r \in (x^{r+1}, y^{r+1})(x^{r+1}, y^{r+1}, x^r y)^{r-1}$ for all $x, y \in A$.

Then $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$.

Proof. Implication $(a) \Rightarrow (b)$ follows from the definitions. Implication $(b) \Rightarrow (c)$ holds in general: if I is generated by x_1, \dots, x_d and if $E(I)_n = 0$, then $(x_1, \dots, x_{d-1})I^{n-1} : x_d^n = (x_1, \dots, x_{d-1})I^{n-2} : x_d^{n-1}$ (Lemma 4.2 [P₂]). Finally, (d) follows from (c) taken $x, y, z \in A$ as $x^{r+1}, y^{r+1}, x^r y$. ■

In order to prove $(d) \Rightarrow \dim A \leq 1$ let us recall some definitions. A set of elements x_1, \dots, x_m of an ideal J of A are called J -independent if every form in $A[T_1, \dots, T_m]$ vanishing at x_1, \dots, x_m has all its coefficients in J . If $I = (x_1, \dots, x_m)$ and $I \subset J$, then x_1, \dots, x_m are J -independent if and only if the natural graded morphism of standard (A/J) -algebras $(A/J)[X_1, \dots, X_m] \rightarrow \mathcal{R}(I) \otimes (A/J)$ is an isomorphism (X_1, \dots, X_m algebraically independent over A/J). If (A, \mathfrak{m}) is noetherian local, then the maximum number of \mathfrak{m} -independent elements in \mathfrak{m} is equal to $\dim A$ [V].

Proposition 4.2 *Let A be a noetherian ring. If there exists an integer $r \geq 1$ such that $(x^r y)^r \in (x^{r+1}, y^{r+1})(x^{r+1}, y^{r+1}, x^r y)^{r-1}$ for all $x, y \in A$, then $\dim A \leq 1$.*

Proof. Since the hypothesis localizes, we may assume (A, \mathfrak{m}, k) is a noetherian local ring. Suppose $\dim A \geq 2$. Then there exists two \mathfrak{m} -independent elements x, y . In particular, if $I = (x, y)$, $\bar{\alpha} : k[X, Y] \rightarrow \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ defined by $\bar{\alpha}(X) = x + \mathfrak{m}I^2$, $\bar{\alpha}(Y) = y + \mathfrak{m}I^2$ is a graded isomorphism of standard k -algebras. By hypothesis $(x^r y)^r \in (x^{r+1}, y^{r+1})(x^{r+1}, y^{r+1}, x^r y)^{r-1}$ which is generated by the elements $x^{(i+1)(r+1)+lr} y^{j(r+1)+l}$, $x^{i(r+1)+lr} y^{(j+1)(r+1)+l}$, $i, j, l \geq 0$, $i+j+l = r-1$. The k -vector space isomorphism $\bar{\alpha}_{r(r+1)}$ assures the membership of $(X^r Y)^r$ into the k -vector space spanned by the elements $X^{(i+1)(r+1)+lr} Y^{j(r+1)+l}$, $X^{i(r+1)+lr} Y^{(j+1)(r+1)+l}$, $i, j, l \geq 0$, $i+j+l = r-1$. Since all of them are elements of a k -basis of $k[X, Y]_{r(r+1)}$, then either $(X^r Y)^r = X^{(i+1)(r+1)+lr} Y^{j(r+1)+l}$ or either $(X^r Y)^r = X^{i(r+1)+lr} Y^{(j+1)(r+1)+l}$, for some $i, j, l \geq 0$, $i+j+l = r-1$. But, it is not difficult to see that there are not integers $i, j, l \geq 0$ verifying any of both equations. ■

Remark 4.3 The underlying idea in the proof of Proposition 4.2 is that for any two \mathfrak{m} -independent elements x, y of A , do not exist r -relations $T_1 f(T_1, T_2, T_3) + T_2 g(T_1, T_2, T_3) - T_3^r$, f, g forms of degree $r-1$, among the three ordered elements $x^{r+1}, y^{r+1}, x^r y$. In particular, $T_1^r T_2 - T_3^{r+1}$ must be an effective $(r+1)$ -relation among the three ordered elements $x^{r+1}, y^{r+1}, x^r y$ (since any form of degree r dividing $T_1^r T_2 - T_3^{r+1}$ should contain T_3^r as an additive factor).

Remark 4.4 There exist (necessarily non noetherian) local rings with $\text{grt}(A) < \infty$, but $\dim A \geq 2$. For example, a valuation ring A is Prüfer, thus $\text{grt}(A) = 1$ [Cos], [P₁], but its dimension is not necessarily 1 or less.

5 Artinian modules have finite global relation type

Proposition 5.1 *Let A be a commutative ring, I an ideal of A and M an A -module. If $I^s M = 0$ for some $s \geq 1$, then $\text{rt}(I; M) \leq s$. If $\text{rt}(I) = 1$ and I is finitely generated, then $I \neq 0$ if and only if $I^s \neq 0$ for all $s \geq 1$. If (A, \mathfrak{m}) is artinian local, then $\text{grt}(M) < \infty$ and $\text{grt}(A) = 1$ if and only if A is a field. If A is an artinian ring, then $\text{grt}(M) < \infty$.*

Proof. If $I^s M = 0$ and $\varphi : G \rightarrow \mathcal{R}(I; M)$ is a symmetric presentation of $\mathcal{R}(I; M)$, then $\ker \varphi_n = G_n$ for all $n \geq s$. Thus, for all $n \geq s+1$, $E(I)_n = G_n/V_1 G_{n-1} = 0$ and $\text{rt}(I; M) \leq s$. In order to prove the second assertion we may suppose that (A, \mathfrak{m}, k) is local. If $\text{rt}(I) = 1$, the natural graded morphism of standard k -algebras $\mathbf{S}^k(I/\mathfrak{m}I) \rightarrow \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ is an isomorphism. If $I \neq 0$, then $\mathbf{S}^k(I/\mathfrak{m}I)$ is a polynomial ring in $\mu(I)$ variables, thus $I^s/\mathfrak{m}I^s \neq 0$ for all $s \geq 1$ and $I^s \neq 0$ since I is finitely generated. If (A, \mathfrak{m}) is artinian local, there exists an integer $s \geq 1$, such that $I^s M \subseteq \mathfrak{m}^s M = 0$ for every ideal I of A . Thus $\text{grt}(M) \leq s$. Moreover, if $\text{grt}(A) = 1$, then $\text{rt}(\mathfrak{m}) = 1$ and $\mathfrak{m}^s = 0$. Hence $\mathfrak{m} = 0$ and A is a field. If A is artinian, it has a finite number of maximal ideals and since $\text{grt}(M) = \sup\{\text{grt}(M_{\mathfrak{m}}) \mid \mathfrak{m} \in \text{Max}(A)\}$, then $\text{grt}(M) < \infty$. ■

Remark 5.2 The minimum integer $s \geq 1$ such that $I^s = 0$, for a nilpotent ideal I , is not necessarily equal to its relation type. For example, take $I = (x, y) \subset A = k[[X, Y]]/(X^n, Y^n)$, where x, y denote the classes of X, Y in A . Then $I^{2n-1} = 0$, $I^{2n-2} \neq 0$ and $\text{rt}(I) = n$. Indeed, since $yI^{n-2} : x^{n-1} \subsetneq yI^{n-1} : x^n = A$, then $E(I)_n \neq 0$ and $\text{rt}(I) \geq n$. Moreover, since $(0 : y) \cap I^{p-1} = (x^{p-n} y^{n-1}) = x((0 : y) \cap I^{p-2})$ for all $p \geq n+1$, then $E(I)_p = 0$ for all $p \geq n+1$ and $\text{rt}(I) \leq n$ (Proposition 4.5 [P₂]).

Remark 5.3 There exist (necessarily non noetherian) local rings with $\dim A = 0$, but $\text{grt}(A) = \infty$. For example, $A = k[T_1, \dots, T_m, \dots]/(T_1^2, \dots, T_m^{m+1}, \dots)$, with k a field, is a zero dimensional local ring. If t_m denotes the residue class of T_m , $(0 : t_m^m) \subsetneq (0 : t_m^{m+1}) = A$ and $\text{rt}((t_m)) = m + 1$.

6 Proof of Theorem 3 in the local case

We first need to reduce to Cohen-Macaulay local rings and modules.

Lemma 6.1 *Let A be a noetherian ring, I an ideal of A and $N \subseteq M$ two finitely generated A -modules such that $I^t N = 0$ for a certain integer $t \geq 1$. Then $\text{rt}(I; M) \leq \text{rt}(I; M/N) + t$. In particular, if I, J are two ideals of A such that $I^t J = 0$ for a certain integer $t \geq 1$, then $\text{rt}(I) \leq \text{rt}((I+J)/J) + t$.*

Proof. Let $s = s(N, M; \{I\})$ be the strong uniform number for the pair (N, M) with respect to the set of ideals $\{I\}$. If $n \geq s + t$, then $I^n M \cap N = I^{n-s}(I^s M \cap N) \subseteq I^{n-s} N \subseteq I^t N = 0$. Let $F = \mathcal{R}(I; M/N)$, $G = \mathcal{R}(I; M)$ and $\varphi : G \rightarrow F$ defined by $\varphi_n : G_n = I^n M \rightarrow I^n M / I^n M \cap N = I^n M + N/N = F_n$. We have $\ker \varphi_n = I^n M \cap N = 0$ for all $n \geq s + t$. Therefore, using Remark 2.5 and Theorem 2, $\text{rt}(I; M) = \text{rt}(G) \leq \max(\text{rt}(F), s + t) \leq \max(\text{rt}(I; M/N), \text{rt}(I; M/N) + t) = \text{rt}(I; M/N) + t$. ■

Corollary 6.2 *Let (A, \mathfrak{m}) be a noetherian local ring, M a finitely generated A -module and $N \subseteq M$ a submodule of finite length. Then $\text{grt}(M) \leq \text{grt}(M/N) + \text{length}(N)$.*

Proof. If $\text{length}(N) = t$, then $I^t N \subseteq \mathfrak{m}^t N = 0$ for every ideal I of A . Thus, by Lemma 6.1, $\text{rt}(I; M) \leq \text{rt}(I; M/N) + t \leq \text{grt}(M/N) + t$. Taking the supremum over all ideals I of A , we have $\text{grt}(M) \leq \text{grt}(M/N) + t$. ■

Next lemma is a generalization to modules of a well known result for rings (see, for instance, [T₂]).

Lemma 6.3 *Let (A, \mathfrak{m}) be a one dimensional Cohen-Macaulay local ring. Let M be a maximal Cohen-Macaulay module. If I is an \mathfrak{m} -primary ideal of A , $\text{rt}(I; M) \leq e(A)$.*

Proof. Applying the tensor product $- \otimes A[t]_{\mathfrak{m}[t]}$, we may assume that the residue field $k = A/\mathfrak{m}$ is infinite. By Theorem 1.1 in [S], $\mu(I) \leq e(A) = e$ and $\mu(I^e) \leq e < \binom{e+1}{1}$. By Theorem 2.3 in [S], there exists $y_0 \in I$ such that $I^e = y_0 I^{e-1}$. In particular, for all $n \geq e$, $I^n = y_0 I^{n-1}$. Since $\mathfrak{m} \not\subseteq Z(M)$, then $y_0 \notin Z(M)$. Consider the complex of A -modules:

$$\Lambda_2(I) \otimes I^{n-2} M \xrightarrow{\partial_{2,n}} I \otimes I^{n-1} M \xrightarrow{\partial_{1,n}} I^n M \longrightarrow 0,$$

where $\partial_{2,n}((x \wedge y) \otimes z) = y \otimes xz - x \otimes yz$ and $\partial_{1,n}(x \otimes t) = xt$, $x, y \in I$, $z \in I^{n-2} M$ and $t \in I^{n-1} M$. By Proposition 2.6, $E(I; M)_n = \ker \partial_{1,n} / \text{im} \partial_{2,n}$. Let us see $\ker \partial_{1,n} = \text{im} \partial_{2,n}$ for all $n \geq e + 1$. Indeed, take $u = \sum x_i \otimes y_0 z_i \in \ker \partial_{1,n}$, $x_i \in I$, $z_i \in I^{n-2} M$. Then $0 = \partial_{1,n}(u) = y_0 \sum x_i z_i$ and, since $y_0 \notin Z(M)$, $\sum x_i z_i = 0$. Therefore, if $v = \sum (y_0 \wedge x_i) \otimes z_i \in \Lambda_2(I) \otimes I^{n-2} M$, then $\partial_{2,n}(v) = \sum x_i \otimes y_0 z_i - \sum y_0 \otimes x_i z_i = u$. So $E(I; M)_n = 0$ for all $n \geq e + 1$ and $\text{rt}(I; M) \leq e(A)$. ■

Notations 6.4 Let (A, \mathfrak{m}) be a one dimensional noetherian local ring. Denote by $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ the minimal primary components of (0) . If A is Cohen-Macaulay, $(0) = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_s$ is a minimal primary decomposition of (0) . If A is not Cohen-Macaulay, there exist an \mathfrak{m} -primary ideal \mathfrak{q}_{s+1} such that $(0) = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_s \cap \mathfrak{q}_{s+1}$ is a minimal primary decomposition of (0) . Let $n \geq 1$ be the minimum integer such that $\mathfrak{n}(A)^n = 0$. Let $n_i \geq 1$ be the minimum integer such that $\mathfrak{p}_i^{n_i} \subset \mathfrak{q}_i$, $\mathfrak{p}_i = r(\mathfrak{q}_i)$,

$i = 1, \dots, s$. For each $1 \leq i_1 < \dots < i_l \leq s$, denote $t_{i_1, \dots, i_l} = \max\{n_i \mid i \neq i_1, \dots, i_l\}$ and $e(A)$ the multiplicity of A . Finally, set $\text{brt}(A) = \max\{n, e(A/(\mathfrak{q}_{i_1} \cap \dots \cap \mathfrak{q}_{i_l})) + t_{i_1, \dots, i_l} \mid 1 \leq i_1 < \dots < i_l \leq s\}$, which is finite.

Proposition 6.5 *Let (A, \mathfrak{m}) be a one dimensional noetherian local ring and $J = H_{\mathfrak{m}}^0(A)$. Let M be a one dimensional finitely generated A -module and $N = H_{\mathfrak{m}}^0(M)$. Then $\text{grt}(A) \leq \text{brt}(A/J) + \text{length}(J)$ and $\text{grt}(M) \leq \text{brt}(A/J) + \text{length}(N)$. If A and M are Cohen-Macaulay, $\text{grt}(M) \leq \text{brt}(A)$.*

Proof. Since $\text{length}(N) = t < \infty$, by Corollary 6.2, $\text{grt}(M) \leq \text{grt}(M/N) + t$. Since $JM \subseteq N$, then $J \subseteq \text{Ann}_A(M/N)$ and $\text{grt}_A(M/N) = \text{grt}_{A/J}(M/N)$. We thus may assume A is a one dimensional Cohen-Macaulay ring and M is a maximal Cohen-Macaulay module. Let us prove $\text{grt}(M) \leq \text{brt}(A)$. If $I \subset \mathfrak{n}(A)$, $I^n \subset \mathfrak{n}(A)^n = 0$ and $\text{rt}(I; M) \leq n$ (Proposition 5.1). If $I \not\subset \mathfrak{n}(A)$, let $1 \leq i_1 < \dots < i_l \leq s$ be all the subindexes i_j such that $I \not\subset \mathfrak{p}_{i_j}$. Set $J_{i_1, \dots, i_l} = \mathfrak{q}_{i_1} \cap \dots \cap \mathfrak{q}_{i_l}$. Then $I^{t_{i_1, \dots, i_l}} J_{i_1, \dots, i_l} \subseteq \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_s = 0$ and, by Lemma 6.1, $\text{rt}(I; M) \leq \text{rt}(I; M/J_{i_1, \dots, i_l}M) + t_{i_1, \dots, i_l} = \text{rt}((I + J_{i_1, \dots, i_l})/J_{i_1, \dots, i_l}; M/J_{i_1, \dots, i_l}M) + t_{i_1, \dots, i_l}$. But, $(I + J_{i_1, \dots, i_l})/J_{i_1, \dots, i_l}$ is an $\mathfrak{m}/J_{i_1, \dots, i_l}$ -primary ideal of the one dimensional Cohen-Macaulay local ring $A/J_{i_1, \dots, i_l}$ and $M/J_{i_1, \dots, i_l}M$ is a maximal Cohen-Macaulay module. Therefore, by Lemma 6.3, $\text{rt}((I + J_{i_1, \dots, i_l})/J_{i_1, \dots, i_l}; M/J_{i_1, \dots, i_l}M) \leq e(A/J_{i_1, \dots, i_l})$. ■

Example 6.6 Let (A, \mathfrak{m}) be a one dimensional noetherian local ring. If A is reduced, then $\text{grt}(A) \leq e(A) + 1$. If A is a domain, then $\text{grt}(A) \leq e(A)$.

Proof. Since A is Cohen-Macaulay, by Proposition 6.5, $\text{grt}(A) \leq \text{brt}(A)$. Following the notations in 6.4, if A is reduced, $n = n_1 = \dots = n_s = 1$, $t_{i_1, \dots, i_l} = 1$ for all $(i_1, \dots, i_l) \neq (1, \dots, s)$ and $t_{1, \dots, s} = 0$. Since $e(A/J) \leq e(A)$, then $\text{brt}(A) \leq e(A) + 1$. If A is a domain, then $n = 1$, $n_1 = 1$, $t_1 = 0$ and $\text{brt}(A) = e(A)$. ■

Example 6.7 Let k be a field and $g \geq 1$ an integer. Set $R = k[t^{g+1}, t^{g+2}, \dots, t^{2g+1}] \subset k[t]$ (t a variable over k), $\mathfrak{n} = (t^{g+1}, t^{g+2}, \dots, t^{2g+1})$, $A = R_{\mathfrak{n}}$ and $\mathfrak{m} = \mathfrak{n}R_{\mathfrak{n}}$. Then (A, \mathfrak{m}, k) is a one dimensional noetherian local domain and $\text{grt}(A) = e(A) = g + 1$.

Proof. By Example 6.6, $\text{grt}(A) \leq e(A)$. For all $n \geq 1$, $\mathfrak{m}^n = (t^{(g+1)n}, \dots, t^{(g+1)n+g})$, $\mu(\mathfrak{m}^n) = g + 1$ and $e(A) = g + 1$. For $n \geq 2$, take $I = (t^{g+1}, t^{g+2})$ and $J = {}_{g, n-1}J = t^{g+1}I^{n-2} : t^{(g+2)(n-1)}$. Remark that $J_{g, n-1} \subseteq J_{g, n}$ and that $E(I)_n = 0$ if and only if $J_{g, n-1} = J_{g, n}$ (Proposition 4.5 [P₂]). If $g = 1$, then $I = \mathfrak{m}$, $E(I)_2 \neq 0$ and $2 \leq \text{rt}(I) \leq e(A) = 2$. Suppose $g \geq 2$. Then $J_{g, 1} = t^{g+1} : t^{g+2} = \mathfrak{m}$. Moreover, $t^{(g+2)g} \notin t^{g+1}I^{g-1}$ and $\mathfrak{m} \subseteq J_{g, g} \subsetneq A$. Moreover $J_{g, g+1} = A$. Thus, $E(I)_n = 0$ for all $2 \leq n \leq g$ and $E(I)_{g+1} \neq 0$. Hence $g + 1 \leq \text{rt}(I) \leq e(A) = g + 1$, $\text{rt}(I) = g + 1$ and $\text{grt}(A) = g + 1$. Remark that $\mathfrak{m}^n = t^{g+1}\mathfrak{m}^{n-1}$ for all $n \geq 2$. So the reduction number of \mathfrak{m} is $\text{rn}(\mathfrak{m}) = 1$ and $1 < \text{rt}(\mathfrak{m}) \leq \text{rn}(\mathfrak{m}) + 1 = 2$ [T₂] while $\text{grt}(A) = g + 1$. ■

Example 6.8 Let k be a field, $a \geq 1$ a positive integer and $A = k[[X, Y]]/(X^a Y)$. Then A is a one dimensional complete intersection local ring with $\text{grt}(A) = \text{brt}(A) = a + 1$.

Proof. By Proposition 6.5, $\text{grt}(A) \leq \text{brt}(A)$. Let x, y denote the residue classes of X, Y and let $\mathfrak{m} = (x, y)$ be the maximal ideal of A . Since $\mu(\mathfrak{m}^n) = a + 1$ for all $n \geq a$, the multiplicity of A is $e(A) = a + 1$. The minimal primary decomposition of A is $(0) = \mathfrak{q}_1 \cap \mathfrak{q}_2$, $\mathfrak{q}_1 = (x^a)$, $\mathfrak{q}_2 = (y)$. Following the notations in 6.4, $\mathfrak{p}_1 = (x)$, $\mathfrak{p}_2 = (y)$, $\mathfrak{n}(A) = (xy)$, $n = n_1 = a$, $n_2 = 1$, $t_1 = n_2 = 1$, $t_2 = n_1 = a$, $t_{1,2} = 0$. Moreover, $A/\mathfrak{q}_1 = k[[X, Y]]/(X^a)$ and $e(A/\mathfrak{q}_1) = a$; $A/\mathfrak{q}_2 = k[[X]]$ and $e(A/\mathfrak{q}_2) = 1$. Therefore, $\text{brt}(A) = a + 1$. On the other hand, $x((0 : y) \cap \mathfrak{m}^{a-1}) = (x^{a+1}) \subsetneq (x^a) = (0 : y) \cap \mathfrak{m}^a$. Thus $E(\mathfrak{m})_{a+1} \neq 0$ and $\text{rt}(\mathfrak{m}) \geq a + 1$ (Proposition 4.5 [P₂]). ■

7 Final proofs

Lemma 7.1 Let (A, \mathfrak{m}) be a one dimensional Cohen-Macaulay local ring with a unique minimal prime \mathfrak{p} and let $n \geq 1$ be such that $\mathfrak{p}^n = 0$. If M is a maximal Cohen-Macaulay A -module, then $\text{grt}(M) \leq \max\{n, e(A)\} = \text{brt}(A)$. Moreover, if A/\mathfrak{p} is a discrete valuation ring, then $\text{grt}(M) \leq \max\{n, \sum_{i=0}^{n-1} \mu(\mathfrak{p}^i)\}$.

Proof. By Proposition 6.5, $\text{grt}(M) \leq \text{brt}(A)$. If $I \subseteq \mathfrak{p}$, then $I^n \subseteq \mathfrak{p}^n = 0$ and $\text{rt}(I; M) \leq n$. If $I \not\subseteq \mathfrak{p}$, then I is an \mathfrak{m} -primary ideal of a one dimensional Cohen-Macaulay local ring. Hence, by Lemma 6.3, $\text{rt}(I; M) \leq e(A)$. Remark that $\text{brt}(A) = \max\{n, e(A)\}$. If moreover, A/\mathfrak{p} is a discrete valuation ring, there exists $u \in A$ such that $\mathfrak{m} = uA + \mathfrak{p}$. Thus, for $r \geq n$, $\mathfrak{m}^r = \sum_{i=0}^{n-1} u^{r-i} \mathfrak{p}^i$ and for $r \gg 1$, $e(A) = \mu(\mathfrak{m}^r) = \mu(\sum_{i=0}^{n-1} u^{r-i} \mathfrak{p}^i) \leq \sum_{i=0}^{n-1} \mu(\mathfrak{p}^i)$. ■

Example 7.2 Let k be a field, $a, b \geq 1$ two positive integers and $A = k[[X, Y]]/(X^a, X^b Y)$. Then A is a one dimensional noetherian local ring with $\text{grt}(A) = a$. Moreover, if $a \leq b$, then $J = H_{\mathfrak{m}}^0(A) = 0$ and $\text{brt}(A) = a$. If $a > b$, then $J = H_{\mathfrak{m}}^0(A) \neq 0$ and $\text{brt}(A/J) + \text{length}(J) = a + b$.

Proof. Let x, y denote the residue classes of X, Y and let $\mathfrak{m} = (x, y)$ be the maximal ideal of A . Remark that $\text{rt}((x)) = a \leq \text{grt}(A)$. If $a \leq b$, $A = k[[X, Y]]/(X^a)$ is a one dimensional Cohen-Macaulay ring with the unique minimal prime (x) . By Lemma 7.1, $a \leq \text{grt}(A) \leq \text{brt}(A) = \max\{a, e(A)\} = a$. If $a > b$, then $I(x^{a-1}) \subseteq (x, y)(x^{a-1}) \subseteq (x^a, x^b y) = 0$. By Lemma 6.1, $\text{rt}(I) \leq \text{rt}((I + (x^{a-1})) / (x^{a-1})) + 1 \leq \text{grt}(A/(x^{a-1})) + 1$. But, $A/(x^{a-1}) = k[[X, Y]]/(X^{a-1}, X^b Y)$. Repeating the same argument, we get $a \leq \text{grt}(A) \leq \text{grt}(A/(x^{a-(a-b)})) + (a - b) = \text{grt}(k[[X, Y]]/(X^b)) + (a - b) = b + (a - b) = a$. On the other hand, $J = H_{\mathfrak{m}}^0(A) = (0 : \mathfrak{m}^{a-b}) = (x^b)$ and $\text{length}(J) = a - b$. $A/J = k[[X, Y]]/(X^b)$ and, as before, $\text{brt}(A/J) = b$. Thus, $\text{brt}(A/J) + \text{length}(J) = a + b$. ■

Theorem 3 Let A be an excellent (in fact $J - 2$) ring. The following conditions are equivalent:

- (i) $\text{grt}(M) < \infty$ for all finitely generated A -module M .
- (ii) $\text{grt}(A) < \infty$.
- (iii) There exists an $r \geq 1$ such that $\text{rt}(I) \leq r$ for every three-generated ideal I of A .
- (iv) There exists an $r \geq 1$ such that $(x^r y)^r \in (x^{r+1}, y^{r+1})(x^{r+1}, y^{r+1}, x^r y)^{r-1}$ for all $x, y \in A$.
- (v) $\dim A \leq 1$.

Proof. Implications (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) are obvious, (iii) \Rightarrow (iv) is Remark 4.1 and (iv) \Rightarrow (v) is Proposition 4.2. Let us prove (v) \Rightarrow (i). Let A be an excellent ring of $\dim A \leq 1$ and let M be a finitely generated A -module. If $\dim M = 0$, then, by Proposition 5.1, $\text{grt}(M) = \text{grt}_{A/\text{Ann}_A(M)}(M) < \infty$. Therefore, we may assume $\dim A = 1$ and $\dim M = 1$. Let $\text{Min}(A) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ be the set of minimal primes of A and let $\text{Ass}(A) = \text{Min}(A) \cup \{\mathfrak{m}_1, \dots, \mathfrak{m}_s\}$, $\mathfrak{m}_i \in \text{Max}(A)$, be the set of associated primes of A . Since $\text{Ass}(A)$ is finite, by Propositions 5.1 and 6.5, $\alpha = \max\{\text{grt}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Ass}(A)\} < \infty$. Analogously, $\alpha' = \max\{\text{grt}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Ass}(M)\} < \infty$. If $r \geq 2$, and for each $1 \leq i_1 < \dots < i_l \leq r$ with $l \geq 2$, consider $\Gamma_{i_1, \dots, i_l} = V(\mathfrak{p}_{i_1} + \dots + \mathfrak{p}_{i_l})$ and $\Gamma = \bigcup_{1 \leq i_1 < \dots < i_l \leq r} \Gamma_{i_1, \dots, i_l}$. If $r = 1$, take $\Gamma = \emptyset$. In any case, Γ is a closed finite subset of $\text{Spec}(A)$. By Proposition 6.5, $\gamma = \max\{\text{grt}(M_{\mathfrak{m}}) \mid \mathfrak{m} \in \Gamma\} < \infty$. Let $\Sigma = \text{Sing}(A/\mathfrak{p}_1) \cup \dots \cup \text{Sing}(A/\mathfrak{p}_r)$, $\text{Sing}(A/\mathfrak{p}_i) = \{\mathfrak{m} \in \text{Max}(A) \mid \mathfrak{m} \supset \mathfrak{p}_i \text{ and } A_{\mathfrak{m}}/\mathfrak{p}_i A_{\mathfrak{m}} \text{ is not regular}\}$. By hypothesis, $\text{Sing}(A/\mathfrak{p}_i)$ is a closed subset of $\text{Spec}(A)$. In particular, $\text{Sing}(A/\mathfrak{p}_i)$ and Σ are finite. Again by Proposition 6.5, $\sigma = \max\{\text{grt}(M_{\mathfrak{m}}) \mid$

$\mathfrak{m} \in \Sigma\} < \infty$. Now, take $\mathfrak{m} \in \text{Max}(A)$, $\mathfrak{m} \notin \text{Ass}(A) \cup \text{Ass}(M) \cup \Gamma \cup \Sigma$. Thus, $A_{\mathfrak{m}}$ is a one dimensional Cohen-Macaulay local ring, $M_{\mathfrak{m}}$ is a maximal Cohen-Macaulay $A_{\mathfrak{m}}$ -module, \mathfrak{m} contains exactly one minimal prime $\mathfrak{p} \in \text{Min}(A)$, $\mathfrak{m} \supset \mathfrak{p}$, and $A_{\mathfrak{m}}/\mathfrak{p}A_{\mathfrak{m}}$ is a discrete valuation ring. Since A is noetherian, there exists an integer $n \geq 1$ such that $\mathfrak{n}(A)^n = 0$. Thus $\mathfrak{p}^n A_{\mathfrak{m}} = 0$. By Lemma 7.1, $\text{grt}(M_{\mathfrak{m}}) \leq \max\{n, \sum_{i=0}^{n-1} \mu(\mathfrak{p}^i A_{\mathfrak{m}})\} \leq \max\{n, \sum_{i=0}^{n-1} \mu(\mathfrak{p}^i)\}$. If $\mu = \sum_{i=0}^{n-1} \mu(\mathfrak{p}^i)$, then $\text{grt}(M) = \sup\{\text{grt}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Spec}(A)\} \leq \max\{n, \mu, \alpha, \alpha', \gamma, \sigma\} < \infty$. ■

Remark 7.3 There exists (necessarily non $J - 2$) noetherian rings with $\dim A \leq 1$, but $\text{grt}(A) = \infty$. For example, take k a field and $R = k[t_1^2, t_1^3, t_2^3, t_2^4, t_2^5, \dots, t_g^{g+1}, t_g^{g+2}, \dots, t_g^{2g+1}, \dots]$. The ideals $\mathfrak{p}_g = (t_g^{g+1}, t_g^{g+2}, \dots, t_g^{2g+1})$ are prime of height 1. Let S be the multiplicative closed set $S = R - \cup \mathfrak{p}_g$ and $A = S^{-1}R$. Let $\mathfrak{m}_g = S^{-1}\mathfrak{p}_g$. Since all prime ideals of R contained in $\cup \mathfrak{p}_g$ are contained in some \mathfrak{p}_g , then A is a one dimensional noetherian domain with maximal ideals \mathfrak{m}_g [SV]. By Example 6.7, $\text{grt}(A_{\mathfrak{m}_g}) = g + 1$. Thus $\text{grt}(A) = \infty$. Remark $\text{Sing}(A) = \text{Spec}(A) - \{(0)\}$, so A is not $J - 2$.

Theorem 1 *Let A be an excellent (in fact $J - 2$) ring and let $N \subseteq M$ be two finitely generated A -modules such that $\dim(M/N) \leq 1$. Then there exists an integer $s \geq 1$ such that, for all integers $n \geq s$ and for all ideals I of A ,*

$$I^n M \cap N = I^{n-s}(I^s M \cap N).$$

Proof. Since $\text{grt}_A(M/N) = \text{grt}_{A/J}(M/N)$ for $J = \text{Ann}_A(M/N)$, we can suppose that A is an excellent ring of $\dim A \leq 1$. Thus, by Theorems 2 and 3, $s(N, M) \leq \text{grt}(M/N) < \infty$. ■

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References

- [Coh] I.S. Cohen: *Commutative rings with restricted minimum condition*. Duke Math. J. **17** (1950), 27-42.
- [Cos] D. Costa: *On the Torsion-Freeness of the Symmetric Powers of an Ideal*. J. Algebra **80** (1983), 152-158.
- [DO] A.J. Duncan, L. O'Carroll: *A full uniform Artin-Rees theorem*. J. reine angew. Math. **394** (1989), 203-207.
- [EH] D. Eisenbud, M. Hochster: *A Nullstellensatz with nilpotents and Zariski's main lemma on holomorphic functions*. J. Algebra **58** (1979), 157-161.
- [HSV] J. Herzog, A. Simis, W.V. Vasconcelos: *Koszul homology and blowing-up rings*. Lecture Notes in Pure and Appl. Math. **84** (1982), pp. 79-169. Marcel Dekker, New York.
- [H₁] C. Huneke: *Uniform bounds in noetherian rings*. Invent. Math. **107** (1992), 203-223.
- [H₂] C. Huneke: *Tight Closure and Its Applications*. CBMS Lecture Notes **88** (1996). American Mathematical Society, Providence.
- [L] Y.H. Lai: *On the relation type of systems of parameters*. J. Algebra **175** (1995), 339-358.
- [O₁] L. O'Carroll: *A uniform Artin-Rees theorem and Zariski's main lemma on holomorphic functions*. Invent. Math. **90** (1987), 674-682.
- [O₂] L. O'Carroll: *A note on Artin-Rees numbers*. Bull. London Math. Soc. **23** (1991), 209-212.

- [P₁] F. Planas Vilanova: *Rings of weak dimension one and syzygetic ideals*. Proc. Amer. Math. Soc. **124** (1996), 3015-3017.
- [P₂] F. Planas Vilanova: *On the module of effective relations of a standard algebra*. Math. Proc. Camb. Phil. Soc. **124** (1998), 215-229.
- [S] J.D. Sally: *Number of generators of ideals in local rings*. Lecture Notes in Pure and Appl. Math. **35** (1978). Marcel Dekker, New York.
- [SV] J.D. Sally, W.V. Vasconcelos: *Stable rings*. J. Pure and Appl. Alg. **4** (1974), 319-336.
- [T₁] N.V. Trung: *Absolutely superficial elements*. Math. Proc. Camb. Phil. Soc. **93** (1983), 35-47.
- [T₂] N.V. Trung: *Reduction exponent and degree bound for the defining equations of graded rings*. Proc. Amer. Math. Soc. **101** (1987), 229-236.
- [V] G. Valla: *Elementi indipendenti rispetto ad un ideale*. Rend. Sem. Mat. Univ. Padova **44** (1970), 339-354.
- [W₁] H.J. Wang: *Some Uniform Properties of 2-Dimensional Local Rings*. J. Algebra **188** (1997), 1-15.
- [W₂] H.J. Wang: *The Relation-Type Conjecture holds for rings with finite local cohomology*. Comm. in Algebra **25** (1997), 785-801.